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Proof of certain Diophantine conjectures and identification of remarkable classes of orthogonal polynomials

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Abstract
Certain *Diophantine* conjectures are proven, and to do so certain remarkable classes of orthogonal polynomials are identified, yielding additional *Diophantine* findings.

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1. Introduction and main results

Recently certain *Diophantine* conjectures were proffered [1, 2] (or see appendix C of [3]). They were arrived at via the investigation of the behaviour of certain *isochronous* many-body problems in the neighbourhood of their equilibrium configurations. In this paper we prove these conjectures, and in order to do so we identify a class of polynomials satisfying three-term recursion relations—hence belonging to an orthogonal class—which seem of interest in their own right, at least inasmuch as they also yield *additional Diophantine* findings. We moreover show how the approach used in this paper provides an alternative proof, and an extension, of *Diophantine* results proffered over one and a half century ago by Sylvester [4] and revisited quite recently by Askey [5] and by Holtz [6].

The first of these conjectures—which coincides, up to trivial notational changes (and the correction of a trivial misprint), with conjecture 4.1 of [1]—states that the *tridiagonal* $N \times N$ matrix $U(N)$ defined componentwise as follows,

$$U_{n,n}(N) = N(N-1) - (n-1)^2 - (N-n)^2 = -2n^2 + (N+1)(2n-1), \qquad n = 1, \dots, N, \tag{1a}$$

$$U_{n,n-1}(N) = (n-1)^2, \qquad n = 2, \dots, N, \tag{1b}$$

$$U_{n,n+1}(N) = (N-n)^2, \qquad n = 1, \ldots, N-1, \tag{1c}$$

(of course with all other elements vanishing) has the N eigenvalues $n(n-1)$, $n = 1, \ldots, N$, i.e.

$$\det[x - U(N)] = \prod_{n=1}^{N} [x - n(n-1)]. \tag{2}$$

Note the symmetry property of this $N \times N$ matrix $U(N)$ under the exchange $n \Longleftrightarrow N+1-n$. And, more importantly, also note that the argument N of this matrix $U(N)$ plays a double role in its definition: it denotes the *order* of this matrix, and moreover it appears as a *parameter* in its definition.

To prove this conjecture we firstly introduce the (more general) *tridiagonal* $M \times M$ matrix $V(M, \nu)$ defined componentwise as follows:

$$V_{m,m}(M, \nu) = \nu(\nu-1) - (m-1)^2 - (\nu-m)^2 = -2m^2 + (\nu+1)(2m-1), \quad m = 1, \ldots, M, \tag{3a}$$

$$V_{m,m-1}(M, \nu) = (m-1)^2, \qquad m = 2, \ldots, M, \tag{3b}$$

$$V_{m,m+1}(M, \nu) = (\nu-m)^2, \qquad m = 1, \ldots, M-1, \tag{3c}$$

(of course with all other elements vanishing), and the class of polynomials

$$p_n^{(\nu)}(x) = \det[x - V(n, \nu)]. \tag{4}$$

It is easily seen that $p_n^{(\nu)}(x)$ is a monic polynomial of degree n in the variable x and it is also a polynomial of degree n in the (*a priori* arbitrary) parameter ν, and that

$$p_1^{(\nu)}(x) = x - \nu + 1, \qquad p_2^{(\nu)}(x) = x^2 - 2(2\nu-3)x + 2(\nu-1)(\nu-2). \tag{5}$$

It is as well plain (see (1) and (3)) that

$$V(n, n) = U(n), \tag{6}$$

hence the conjecture reported above, see (2), amounts to formula (see (4) and (6))

$$p_n^{(n)}(x) = \prod_{m=1}^{n} [x - m(m-1)], \tag{7a}$$

which for future reference is complemented by the assignment

$$p_0^{(0)}(x) = 1. \tag{7b}$$

In the following section 2 we show that the polynomials $p_n^{(\nu)}(x)$ satisfy (and are in fact defined by) the three-term recursion relation

$$p_{n+1}^{(\nu)}(x) = [x + 2n^2 - 2(\nu-1)n - \nu + 1]p_n^{(\nu)}(x) - n^2(n-\nu)^2 p_{n-1}^{(\nu)}(x), \qquad n = 1, 2, \ldots \tag{8a}$$

with the 'initial assignments'

$$p_{-1}^{(\nu)}(x) = 0, \qquad p_0^{(\nu)}(x) = 1. \tag{8b}$$

This is in fact a rather trivial consequence of their definition (4) with (3)—and, as we show in the following section, it entails for these polynomials the rather explicit representation

$$p_n^{(\nu)}(x) = p_n^{(n)}(x) + \sum_{m=0}^{n-1} \left\{ \left(\frac{n!}{m!} \right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\}. \tag{9}$$

On the right-hand side of this formula the expressions $p_n^{(n)}(x)$ (and of course likewise $p_m^{(m)}(x)$) are *a priori* given by (7) (see the derivation of this formula in the following section), but clearly setting $\nu = n$ in this formula (9) this notation gets validated because the sum on the right-hand side of this formula disappears due to the vanishing of the product appearing in it: namely the conjecture (7*a*) is thereby proven.

Remark. In formula (9) the first term on the right-hand side could be absorbed in the sum following it, by extending that sum to n rather than $n-1$, with the standard assumption that a product has *unit* value whenever the lower limit in the range of its index exceeds the upper limit. An analogous observation applies to several formulae written below. We prefer the more explicit notation employed above, and below, because we consider it is more transparent in the context of our treatment.

Clearly this expression, (9) with (7), of the polynomial $p_n^{(\nu)}(x)$, which is valid when n is a *positive* integer and ν is an *arbitrary* number, entails the relation

$$p_n^{(j)}(x) = p_n^{(n)}(x) + \sum_{m=j}^{n-1} \left\{ \left(\frac{n!}{m!}\right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\}, \tag{10}$$

provided j is a *positive* integer not exceeding n, $j \leqslant n$ (since the last product on the right-hand side of (9) vanishes for $m < j \leqslant n$). Moreover, since clearly (see (7*a*))

$$p_m^{(m)}[k(k-1)] = 0 \qquad \text{for} \quad k = 1, 2, \ldots, m, \tag{11}$$

(9) with (7) entails (provided k is a *positive integer*)

$$p_n^{(\nu)}[k(k-1)] = p_n^{(n)}[k(k-1)] + \sum_{m=0}^{\min[n-1,k-1]} \left\{ \left(\frac{n!}{m!}\right)^2 \frac{p_m^{(m)}[k(k-1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} \tag{12a}$$

and (see (10))

$$p_n^{(j)}[k(k-1)] = p_n^{(n)}[k(k-1)] + \sum_{m=j}^{\min[n-1,k-1]} \left\{ \left(\frac{n!}{m!}\right)^2 \frac{p_m^{(m)}[k(k-1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\}. \tag{12b}$$

Hence (see (11), and note that the sum on the right-hand side of (12*b*) vanishes if its lower limit exceeds its upper limit)

$$p_n^{(j)}[k(k-1)] = 0 \qquad \text{if} \quad k = 1, 2, \ldots, j, \qquad j = 1, 2, \ldots, n. \tag{13}$$

In words, the polynomial $p_n^{(j)}(x)$ (of degree n in the variable x) with j any *positive integer* not exceeding n ($j = 1$ or 2 or, ..., or n) has the j zeros $k(k-1)$ with $k = 1, \ldots, j$. This is a remarkable *Diophantine* property associated with the family of orthogonal polynomials characterized by the three-term recursion relation (8); it includes the result (7*a*), which is reproduced for $j = n$. Note that these j zeros of the polynomial $p_n^{(j)}(x)$ are independent of its order n.

It is moreover easily seen that (10) with (7) entails

$$p_n^{(n-1)}(x) = (x+n)\, p_{n-1}^{(n-1)}(x) = (x+n) \prod_{m=1}^{n-1} [x - m(m-1)], \tag{14}$$

$$\begin{aligned} p_n^{(n-2)}(x) &= [x^2 + 2(2n-1)x + 2n(n-1)]\, p_{n-2}^{(n-2)}(x) \\ &= [x - x_+(n)]\,[x - x_-(n)] \prod_{m=1}^{n-2} [x - m(m-1)], \end{aligned} \tag{15a}$$

of course with

$$x_{\pm}(n) = -2n + 1 \pm [n^2 + (n-1)^2]^{1/2}. \tag{15b}$$

It is thus seen that, in addition to featuring the $n-1$ integer zeros $k(k-1)$ implied by (13), the polynomial $p_n^{(n-1)}(x)$ also vanishes at $x = -n$,

$$p_n^{(n-1)}(-n) = 0; \tag{16}$$

hence also this polynomial $p_n^{(n-1)}(x)$, as well as the polynomial $p_n^{(n)}(x)$, has the remarkable *Diophantine* property that *all* its n zeros are *integer* numbers (see (14)). On the other hand, as shown by (15), for a generic (*positive integer*) value of $n \geqslant 3$ the polynomial $p_n^{(n-2)}(x)$ has $n-2$ integer zeros (see (13)), but its remaining 2 zeros are generally *not* integer numbers (see (15*b*))—except when n is the second entry in the sequence A001652 [8] of twin Pythagorean triples (we owe this last remark to Christophe Smet).

Analogous results also hold for the conjecture presented in [2]. We now describe these findings; since the treatment is analogous to that reported above, the following presentation is somewhat more terse.

This conjecture [2] states that the *tridiagonal* $N \times N$ matrix $\tilde{U}(N)$ defined componentwise as follows,

$$\tilde{U}_{n,n}(N) = 2n(N+1-n), \qquad n = 1, \ldots, N, \tag{17a}$$

$$\tilde{U}_{n,n-1}(N) = -n(N+1-n), \qquad n = 2, \ldots, N, \tag{17b}$$

$$\tilde{U}_{n,n+1}(N) = -n(N+1-n), \qquad n = 1, \ldots, N-1, \tag{17c}$$

(of course with all other elements vanishing) has the N eigenvalues $n(n+1)$, $n = 1, \ldots, N$, i.e.

$$\det[x - \tilde{U}(N)] = \prod_{n=1}^{N} [x - n(n+1)]. \tag{18}$$

As above, note the symmetry property of this $N \times N$ matrix $\tilde{U}(N)$ under the exchange $n \Longleftrightarrow N+1-n$, as well as the double role played by the number N, which identifies the *order* of this matrix and appears as a *parameter* in its definition.

To prove this conjecture we firstly introduce the (more general) *tridiagonal* $M \times M$ matrix $\tilde{V}(M, \nu)$ defined componentwise as follows:

$$\tilde{V}_{m,m}(M, \nu) = 2m(\nu+1-m), \qquad m = 1, \ldots, M, \tag{19a}$$

$$\tilde{V}_{m,m-1}(M, \nu) = -m(\nu+1-m), \qquad m = 2, \ldots, M, \tag{19b}$$

$$\tilde{V}_{m,m+1}(M, \nu) = -m(\nu+1-m), \qquad m = 1, \ldots, M-1, \tag{19c}$$

(of course with all other elements vanishing), and the class of polynomials

$$\tilde{p}_n^{(\nu)}(x) = \det[x - \tilde{V}(n, \nu)]. \tag{20}$$

Again, it is easily seen that $\tilde{p}_n^{(\nu)}(x)$ is a monic polynomial of degree n in the variable x and it is also a polynomial of degree n in the (*a priori* arbitrary) parameter ν, and that

$$\tilde{p}_1^{(\nu)}(x) = x - 2\nu, \qquad \tilde{p}_2^{(\nu)}(x) = x^2 - 2(3\nu - 2)x + 6\nu(\nu - 1). \tag{21}$$

It is as well plain (see (17) and (19)) that

$$\tilde{V}(n, n) = \tilde{U}(n), \tag{22}$$

hence the conjecture reported above, see (18), amounts to formula (see (20) and (22))

$$\tilde{p}_n^{(n)}(x) = \prod_{m=1}^{n} [x - m(m+1)], \tag{23a}$$

which for future reference is complemented by the assignment

$$\tilde{p}_0^{(0)}(x) = 1. \tag{23b}$$

It is moreover plain that

$$\tilde{p}_n^{(n)}(0) = (-1)^n n!(n+1)!. \tag{23c}$$

In the following section 2 we show that the polynomials $\tilde{p}_n^{(\nu)}(x)$ satisfy (and are in fact defined by) the three-term recursion relation

$$\tilde{p}_{n+1}^{(\nu)}(x) = [x + 2(n+1)(n-\nu)]\, \tilde{p}_n^{(\nu)}(x) - n(n+1)(n-\nu)(n-1-\nu)\tilde{p}_{n-1}^{(\nu)}(x),$$
$$n = 1, 2, \ldots \tag{24a}$$

with the 'initial assignments'

$$\tilde{p}_{-1}^{(\nu)}(x) = 0, \qquad \tilde{p}_0^{(\nu)}(x) = 1. \tag{24b}$$

This is in fact a rather trivial consequence of their definition (20)—and, as we show in the following section 2, it entails for these polynomials the rather explicit representation

$$\tilde{p}_n^{(\nu)}(x) = \tilde{p}_n^{(n)}(x) + \sum_{m=0}^{n-1} \left\{ \left[\frac{n!(n+1)!}{m!(m+1)!} \right] \frac{\tilde{p}_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\}, \tag{25}$$

where (consistently with this formula) $\tilde{p}_n^{(n)}(x)$ is defined by (23). The conjecture (23*a*) is thereby proven.

Clearly this expression, (25) with (23), of the polynomial $\tilde{p}_n^{(\nu)}(x)$, which is valid when n is a *positive* integer and ν is an *arbitrary* number, entails the relation

$$\tilde{p}_n^{(j)}(x) = \tilde{p}_n^{(n)}(x) + \sum_{m=j}^{n-1} \left\{ \left[\frac{n!(n+1)!}{m!(m+1)!} \right] \frac{\tilde{p}_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\}, \tag{26}$$

provided j is a *positive* integer not exceeding n, $j \leqslant n$ (since the last product on the right-hand side of (25) vanishes for $m < j \leqslant n$). Moreover, since clearly (see (23*a*))

$$\tilde{p}_m^{(m)}[k(k+1)] = 0 \qquad \text{for} \quad k = 1, 2, \ldots, m, \tag{27}$$

(25) with (23) entails (provided k is a *positive integer*)

$$\tilde{p}_n^{(\nu)}[k(k+1)] = \tilde{p}_n^{(n)}[k(k+1)] + \sum_{m=0}^{\min[n-1,k-1]} \left\{ \left[\frac{n!(n+1)!}{m!(m+1)!} \right] \frac{\tilde{p}_m^{(m)}[k(k+1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} \tag{28a}$$

and (see (26))

$$\tilde{p}_n^{(j)}[k(k+1)] = \tilde{p}_n^{(n)}[k(k+1)] + \sum_{m=j}^{\min[n-1,k-1]} \left\{ \left[\frac{n!(n+1)!}{m!(m+1)!} \right] \frac{\tilde{p}_m^{(m)}[k(k+1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\}. \tag{28b}$$

Hence (see (27), and note that the sum on the right-hand side of (28*b*) vanishes if its lower limit exceeds its upper limit)

$$\tilde{p}_n^{(j)}[k(k+1)] = 0 \qquad \text{if} \quad k = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n. \tag{29a}$$

In words, the polynomial $\tilde{p}_n^{(j)}(x)$ (of degree n in the variable x) with j any *positive integer* not exceeding n ($j = 1$ or 2 or, ..., or n) has the j zeros $k(k+1)$ with $k = 1, \ldots, j$. This is a remarkable *Diophantine* property associated with the family of orthogonal polynomials characterized by the three-term recursion relation (24); it includes the result (23a), which is reproduced for $j = n$. Again, let us emphasize that these j zeros of the polynomial $\tilde{p}_n^{(j)}(x)$ are independent of its degree n.

It is moreover easily seen, by inserting (23c) in (26), that

$$\tilde{p}_n^{(j)}(0) = 0 \qquad \text{for} \quad j = 0, 1, \ldots, n-1. \tag{29b}$$

(We owe this remark to Christophe Smet.) It is thus seen (see also (29)) that, for $j = 0, 1, \ldots, n-1$, the polynomial $\tilde{p}_n^{(j)}(x)$ features $j+1$ *integer* numbers among its n zeros. Hence *all* its zeros are *integers* for $j = n-1$ (as well as for $j = n$, see (23a)). Indeed clearly (26) yields

$$\tilde{p}_n^{(n-1)}(x) = x\,\tilde{p}_{n-1}^{(n-1)}(x) = x \prod_{m=1}^{n-1} [x - m(m+1)], \tag{30a}$$

as well as

$$\begin{aligned}\tilde{p}_n^{(n-2)}(x) &= x(x+2n)\,\tilde{p}_{n-2}^{(n-2)}(x)\\ &= x(x+2n) \prod_{m=1}^{n-2} [x - m(m+1)].\end{aligned} \tag{30b}$$

Thus we conclude that, not only the polynomial $\tilde{p}_n^{(n)}(x)$, but as well the polynomials $\tilde{p}_n^{(n-1)}(x)$ and $\tilde{p}_n^{(n-2)}(x)$, have the remarkable *Diophantine* property that *all* their n zeros are *integer* numbers, see (23a), (30a) and (30b). It is moreover easily seen from (26) that

$$\tilde{p}_n^{(n-3)}(x) = x\big[x - \tilde{x}_+^{(n)}\big]\big[x - \tilde{x}_-^{(n)}\big] \prod_{m=1}^{n-3} [x - m(m+1)], \tag{30c}$$

with

$$\tilde{x}_\pm^{(n)} = 1 - 3n \pm (1 + 3n^2)^{1/2}. \tag{30d}$$

Hence $\tilde{p}_n^{(n-3)}(x)$ also features only *integer* zeros if $1 + 3n^2$ is the square of an *integer*, namely for $n = 4, 15, 56, 209\ldots$ (sequence A001353 [8]: we owe this remark to Christophe Smet).

Note that the recursion relations (8a) respectively (24a) entail that the polynomials $p_n^{(\nu)}(x)$ respectively $\tilde{p}_n^{(\nu)}(x)$ are *orthogonal* ('Favard theorem': see [7], and, for instance, p 159 of [9]).

It would of course be sufficient at this stage to simply *verify* that the polynomials given by formulae (9) respectively (25) satisfy the recursion relations (8) respectively (24); but it seems more appropriate to prove these formulae—in the following section 2—via a route that makes clear how they were obtained.

The fact that the three-term recursion relation (8) respectively (24) can be solved explicitly, see (9) respectively (25), and that the two classes of orthogonal polynomials $p_n^{(\nu)}(x)$ respectively $\tilde{p}_n^{(\nu)}(x)$ defined by them feature the *Diophantine* properties (13) (see also (30), and the remark at the end of the paragraph including this equation) respectively (29a) (see also (30)) seems remarkable.

The diligent reader will verify that these polynomials, as given by (9) respectively (25), can be identified with generalized hypergeometric functions [9] as follows:

$$p_n^{(\nu)}(x) = n!(1-\nu)_n\; {}_3F_2\,(-n, m_+(x), m_-(x)\,; 1, 1-\nu; 1), \tag{31a}$$

$$m_{\pm}(x) = \frac{1 \pm (1+4x)^{1/2}}{2}, \tag{31b}$$

or equivalently

$$p_n^{(\nu)}[x(x+1)] = n!(1-\nu)_n \; {}_3F_2(-n, x+1, -x; 1, 1-\nu; 1), \tag{31c}$$

respectively

$$\tilde{p}_n^{(\nu)}(x) = (n+1)!(1-\nu)_n \; {}_3F_2(-n, \tilde{m}_+(x), \tilde{m}_-(x); 2, 1-\nu; 1), \tag{32a}$$

$$\tilde{m}_{\pm}(x) = \frac{3 \pm (1+4x)^{1/2}}{2} = 1 + m_{\pm}(x), \tag{32b}$$

or equivalently

$$\tilde{p}_n^{(\nu)}[x(x-1)] = (n+1)!(1-\nu)_n \; {}_3F_2(-n, 1+x, 2-x; 2, 1-\nu; 1). \tag{32c}$$

Here and below the hypergeometric function is of course defined in the standard manner [9]:

$$_nF_m\,[a_1, \ldots, a_n; c_1, \ldots, c_m; z] = \sum_{\ell=0}^{\infty} \frac{(a_1)_\ell \cdots (a_n)_\ell z^\ell}{(c_1)_\ell \cdots (c_m)_\ell \ell!}, \tag{33a}$$

with the symbol $(a)_\ell$ defined as follows,

$$(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)}, \tag{34}$$

where Γ denotes the standard gamma function [9].

Hence, when the parameter ν is a *positive integer* larger than $n, \nu = N+1, N \geqslant n$, our polynomials $p_n^{(\nu)}(x)$ respectively $\tilde{p}_n^{(\nu)}(x)$ can be related to the 'Dual Hahn polynomials' $R_n(x; \gamma, \delta, N)$ (see for instance [10]) via the formulae

$$p_n^{(N+1)}(x) = n!(-N)_n R_n(x; 0, 0, N) \tag{35}$$

respectively

$$\tilde{p}_n^{(N+1)}(x) = (n+1)!(-N)_n R_n(x-2; 1, 1, N). \tag{36}$$

Note that instead our Diophantine results (see (10) and (26) and the formulae following these two equations) refer to the case when n and N are indeed *non-negative* integers but $N+1 \leqslant n$; and that in these cases these formulae must be interpreted with appropriate care.

Finally we apply our approach to a problem treated quite recently by Richard Askey [5] and Olga Holtz [6]. They revisited an old result by Sylvester [4] and provided a proof of the following determinantal formula:

$$\det[x - S(N)] = \prod_{n=1}^{N} [x - (2m - 1 - N)], \tag{37}$$

where S denotes the *tridiagonal* $N \times N$ matrix defined componentwise as follows:

$$S_{n,n-1}(N) = N + 1 - n, \qquad n = 2, \ldots, N, \tag{38a}$$

$$S_{n,n+1}(N) = n, \qquad n = 1, \ldots, N-1, \tag{38b}$$

with *all* other elements vanishing. Once again, note the symmetry property of this $N \times N$ matrix $S(N)$ under the exchange $n \iff N+1-n$, as well as the double role played by the number N, which identifies the *order* of this matrix and appears as a *parameter* in its definition.

Proceeding in close analogy with the preceding treatment (but somewhat more tersely), we now introduce the *tridiagonal* $M \times M$ matrix $T(M, \nu)$ defined componentwise as follows:

$$T_{m,m-1}(M, \nu) = \nu + 1 - m, \qquad m = 2, \ldots, M, \tag{39a}$$

$$T_{m,m+1}(M, \nu) = m, \qquad m = 1, \ldots, M - 1, \tag{39b}$$

of course with all other elements vanishing, and ν an *a priori* arbitrary parameter. We then define the class of polynomials

$$s_n^{(\nu)}(x) = \det[x - T(n, \nu)], \qquad n = 1, 2, \ldots. \tag{40}$$

Clearly $s_n^{(\nu)}(x)$ is a monic polynomial of degree n in the variable x and it is also a polynomial in the parameter ν of degree $\frac{n}{2}$ if n is *even*, $\frac{n-1}{2}$ if n is *odd*, and

$$s_1^{(\nu)}(x) = x, \qquad s_2^{(\nu)}(x) = x^2 + 1 - \nu. \tag{41}$$

It is as well plain (see (38) and (39)) that

$$T(n, n) = S(n), \tag{42}$$

hence the determinantal identity (37) amounts to the formula

$$s_n^{(n)}(x) = \prod_{m=1}^{n} [x - (2m - n - 1)]. \tag{43}$$

In the following section 2 we show that the polynomials $s_n^{(\nu)}(x)$ satisfy (and are in fact defined by) the three-term recursion relation

$$s_{n+1}^{(\nu)}(x) = x s_n^{(\nu)}(x) + n(n - \nu) s_{n-1}^{(\nu)}(x), \qquad n = 1, 2, \ldots \tag{44a}$$

with the 'initial assignment'

$$s_0^{(\nu)}(x) = 1, \tag{44b}$$

which is clearly sufficient to define uniquely the entire sequence of polynomials $s_n^{(\nu)}(x)$ with $n = 1, 2, \ldots$. Note that this recursion relation entails the parity property

$$s_n^{(\nu)}(-x) = (-1)^n s_n^{(\nu)}(x). \tag{45}$$

The recursion relation (44) is a rather trivial consequence of the definition (40) of the polynomials $s_n^{(\nu)}(x)$—and, as we show in the following section 2, it entails for these polynomials the rather explicit representation

$$s_n^{(\nu)}(x) = \hat{s}_n^{(\nu)}(x) + \sum_{m=0}^{n-1} \left\{ \frac{n!}{m!} \frac{\hat{s}_m^{(\nu)}(x)}{(n - m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} \tag{46a}$$

where by definition

$$\hat{s}_0^{(\nu)}(x) = 1, \qquad \hat{s}_n^{(\nu)}(x) = \prod_{m=1}^{n} [x - (2m - \nu - 1)], \qquad n = 1, 2, \ldots. \tag{46b}$$

Clearly by setting $\nu = n$ in this formula one finds that

$$s_n^{(n)}(x) = \hat{s}_n^{(n)}(x), \tag{47}$$

because for $\nu = n$ the sum on the right-hand side of (46*a*) disappears due to the vanishing of the product appearing in it, and via (46*b*) the determinantal formula (43) is thereby proven, reproducing the Sylvester's finding [4] recently proven by Askey [5] and Holtz [6].

Moreover this expression, (46), of the polynomial $s_n^{(\nu)}(x)$, which is valid when n is a *positive* integer and ν is an *arbitrary* number, entails the relation

$$s_n^{(j)}(x) = \hat{s}_n^{(j)}(x) + \sum_{m=j}^{n-1} \left[\frac{n!}{m!} \frac{\hat{s}_m^{(j)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right], \tag{48}$$

provided j is a *positive* integer not exceeding n, $j \leqslant n$ (since the last product on the right-hand side of (46*a*) vanishes for $m < j \leqslant n$). Since clearly (see (46*b*))

$$\hat{s}_m^{(j)}[2k - j - 1] = 0 \qquad \text{for} \quad k = 1, 2, \ldots, m, \tag{49}$$

(46) moreover entails (provided k is a *positive integer*)

$$s_n^{(j)}(2k - j - 1) = \hat{s}_n^{(j)}(2k - j - 1) + \sum_{m=j}^{\min[n-1,k-1]} \left\{ \frac{n!}{m!} \frac{\hat{s}_m^{(j)}(2k - j - 1)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\}. \tag{50}$$

Hence (because the sum on the right-hand side of this formula vanishes if its lower limit exceeds its upper limit; and taking note of (49))

$$s_n^{(j)}(2k - j - 1) = 0 \qquad \text{if} \quad k = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n. \tag{51}$$

In words, the polynomial $s_n^{(j)}(x)$ (of degree n in the variable x) with j any *positive integer* not exceeding n ($j = 1$ or 2 or... or n) has the j zeros $2k - j - 1$ with $k = 1, \ldots, j$. This is a remarkable *Diophantine* property associated with the family of orthogonal polynomials characterized by the three-term recursion relation (44); it includes the result (43), which is reproduced for $j = n$, when this *Diophantine* property identifies *all* the zeros of the monic polynomial $s_n^{(n)}(x)$, yielding its explicit form (43). Once more let us emphasize that these j zeros of the polynomial $s_n^{(j)}(x)$ are independent of its degree n.

Moreover (48) with (46*b*) entails

$$s_n^{(n-1)}(x) = x \prod_{m=1}^{n-1} [x - (2m - n)], \tag{52}$$

$$s_n^{(n-2)}(x) = (x^2 + n - 1)\hat{s}_{n-2}^{(n-2)}(x)$$
$$= [x - \hat{x}_+(n)][x - \hat{x}_-(n)] \prod_{m=1}^{n-2} [x - (2m - n + 1)], \tag{53a}$$

of course with

$$\hat{x}_\pm(n) = \pm(1 - n)^{1/2}. \tag{53b}$$

It is thus seen that, in addition to featuring the $n - 1$ integer zeros $2k - n$ implied by (51), the polynomial $s_n^{(n-1)}(x)$ also vanishes at $x = 0$,

$$s_n^{(n-1)}(0) = 0; \tag{54}$$

hence also this polynomial $s_n^{(n-1)}(x)$, as well as the polynomial $s_n^{(n)}(x)$, has the remarkable *Diophantine* property that *all* its n zeros are *integer* numbers (see (52)). On the other hand, as shown by (53), for a generic (*positive integer*) value of $n \geqslant 3$ the polynomial $s_n^{(n-2)}(x)$ has $n - 2$ integer zeros (see (51)), but its remaining two zeros are *not* integer numbers, indeed they are *not* even real, see (53*b*).

Clearly the property (51) entails the factorization

$$s_n^{(j)}(x) = \sigma_{n-j}^{(j)}(x) \prod_{k=1}^{j} (x - 2k + j + 1), \qquad j = 1, \ldots, n, \tag{55}$$

with $\sigma_{n-j}^{(j)}(x)$ a monic polynomial of degree $n - j$. We owe to Christophe Smet the remark that this polynomial has parity $n - j$, $\sigma_{n-j}^{(j)}(-x) = (-1)^{n-j} \sigma_{n-j}^{(j)}(x)$, hence it vanishes at $x = 0$ if $n - j$ is *odd*: so that the property (54) is actually a special case of the more general finding

$$s_n^{(n-j)}(0) = 0, \qquad j = 1, 3, 5, \ldots, n-1 \quad \text{or} \quad n. \tag{56}$$

On the other hand the parity property (45) clearly implies that $s_n^{(\nu)}(x)$ vanishes at $x = 0$ if n is *odd*,

$$s_{2j-1}^{(\nu)}(0) = 0, \qquad j = 1, 2, 3, \ldots. \tag{57}$$

Finally, the diligent reader will verify that these polynomials, as given by (46), can be identified with the standard hypergeometric function [9] as follows:

$$s_n^{(\nu)}(x) = (1 - \nu)_n \, {}_2F_1 \left(-n, -\frac{x + \nu - 1}{2}; 1 - \nu; 2 \right). \tag{58}$$

Additional remark. Christophe Smet pointed out—on the basis of numerical evidences—that

$$p_n^{(2n+1)}(x) = \prod_{m=1}^{n} [x - 2m(2m - 1)], \tag{59}$$

so that also these polynomials $p_n^{(2n+1)}(x)$ *only* feature *integer* zeros; and that an analogous straightforward factorization can be given for $\tilde{p}_n^{(2n+1)}(x)$ and $s_n^{(2n+1)}(x)$, entailing that these polynomials share the remarkable *Diophantine* property to feature *only integer* zeros. We plan to revisit these findings in our next paper on these topics [11].

2. Proofs

In this section we prove the results reported in the previous section.

Let us consider firstly the findings connected with the first conjecture [1]. As stated above, the fact that the polynomials defined by (4) with (3) satisfy the recursion relation (8) is an easy consequence of the determinantal definition (4) with (3): to verify it compute $\det[x - V(n + 1, \nu)]$ by multiplying the (only two nonvanishing) elements of the last line (or equivalently of the last column) of this determinant by their adjoint determinants, obtaining thereby the recursion (8*a*), and then check if need be that, for $n = 0$ and $n = 1$, (8) yields (5).

It is then convenient to renormalize the polynomials $p_n^{(\nu)}(x)$ via the definition

$$p_n^{(\nu)}(x) = (n!)^2 q_n^{(\nu)}(x), \tag{60}$$

entailing that the polynomials $q_n^{(\nu)}(x)$ satisfy the three-term recursion relation

$$(n + 1)^2 q_{n+1}^{(\nu)}(x) = [x + 2n^2 - 2(\nu - 1)n - (\nu - 1)] q_n^{(\nu)}(x) - (\nu - n)^2 q_{n-1}^{(\nu)}(x), \qquad n = 1, 2, \ldots, \tag{61a}$$

$$q_{-1}^{(\nu)}(x) = 0, \qquad q_0^{(\nu)}(x) = 1, \qquad q_1^{(\nu)}(x) = x - \nu + 1. \tag{61b}$$

The purpose of this step is to obtain a recursion relation, see (61*a*), in which the index n only enters quadratically (rather than quartically, see (8*a*)).

Next, we introduce the generating function

$$Q(x, \nu; z) = \sum_{n=0}^{\infty} (z+1)^{-n} q_n^{(\nu)}(x). \tag{62}$$

It is then rather easy to verify that the recursion relation (61) entails that this generating function satisfies the second-order ODE

$$(z+1)^2 z^2 Q'' + (z+2-2\nu)(z+1)zQ' + [(\nu-1-x)z + \nu(\nu-1) - x]Q = 0, \tag{63}$$

where (just above, and always below) the appended primes denote differentiations with respect to z. It is also plain that, at large values of $|z|$, via (61b) we get from (62)

$$Q(x, \nu; z) = 1 + \frac{x-\nu+1}{z} + O\left(\frac{1}{|z|^2}\right). \tag{64}$$

To solve the ODE (63) and thereby identify the generating function $Q(x, \nu; z)$, it is convenient to set

$$Q(x, \nu; z) = (z+1)^{1-\nu}\, z^{\nu-1} F(x, \nu; z), \tag{65}$$

entailing that $F(x, \nu; z)$ satisfies the ODE

$$(z+1)z^2 F'' + z^2 F' - xF = 0. \tag{66}$$

It is then plain that, by setting

$$F(x, \nu; z) = \sum_{n=0}^{\infty} C_n(x, \nu) z^{-n}, \tag{67}$$

one gets for the quantities $C_n(x, \nu)$ the recursion relation

$$C_n(x, \nu) = \frac{x - n(n-1)}{n^2} C_{n-1}(x, \nu) \tag{68a}$$

entailing

$$C_n(x, \nu) = \frac{p_n^{(n)}(x)}{(n!)^2} C_0(x, \nu). \tag{68b}$$

Here and hereafter $p_n^{(n)}(x)$ is defined by (7).

Hence we conclude that a solution of (66) is provided by the formula

$$F(x, \nu; z) = C_0(x, \nu) \sum_{n=0}^{\infty} \frac{p_n^{(n)}(x)}{(n!)^2} z^{-n}, \tag{69}$$

with $C_0(x, \nu)$ an arbitrary function of its two arguments. Via (65) this yields for the generating function $Q(x, \nu; z)$ the expression

$$Q(x, \nu; z) = C_0(x, \nu) \left(1 + \frac{1}{z}\right)^{1-\nu} \sum_{n=0}^{\infty} \frac{p_n^{(n)}(x)}{(n!)^2} z^{-n}, \tag{70}$$

entailing at large z (via (7))

$$Q(x, \nu; z) = C_0(x, \nu) \left(1 + \frac{x+1-\nu}{z}\right) + O\left(\frac{1}{|z|^2}\right). \tag{71}$$

Comparing with (64) we therefore conclude that $C_0(x, \nu) = 1$, yielding for $Q(x, \nu; z)$ the final expression

$$Q(x, \nu; z) = \left(1 + \frac{1}{z}\right)^{1-\nu} \sum_{n=0}^{\infty} \frac{p_n^{(n)}(x)}{(n!)^2} z^{-n}. \tag{72}$$

It is on the other hand clear that (62) entails the following integral expression for the polynomials $q_n^{(\nu)}(x)$:

$$q_n^{(\nu)}(x) = (2\pi \mathrm{i})^{-1} \oint \mathrm{d}z (z+1)^{n-1} Q(x, \nu; z), \tag{73}$$

with the integral $\oint$ (see just above and always below) being performed, in the complex z-plane, counterclockwise on a contour enclosing the point $z = -1$ (and not the point $z = 0$). Hence, via (72),

$$q_n^{(\nu)}(x) = \sum_{m=0}^{\infty} \frac{p_m^{(m)}(x)}{(m!)^2} (2\pi \mathrm{i})^{-1} \oint \mathrm{d}z\, z^{\nu-1-m} (z+1)^{n-\nu}. \tag{74}$$

We now use the formula

$$(2\pi \mathrm{i})^{-1} \oint \mathrm{d}z\, z^{\nu-1-m} (z+1)^{n-\nu} = \frac{\prod_{\ell=m+1}^{n} (\ell - \nu)}{(n-m)!}, \tag{75}$$

which is easily proven by expanding $z^{\nu-1-m}$ in inverse powers of $(z+1)$,

$$\begin{aligned} z^{\nu-1-m} &= (z+1)^{\nu-1-m} \left(1 - \frac{1}{z+1}\right)^{\nu-1-m} \\ &= (z+1)^{\nu-1-m} \sum_{j=0}^{\infty} (-1)^j \binom{\nu-1-m}{j} (z+1)^{-j}. \end{aligned} \tag{76}$$

We thereby obtain the following expression of the polynomials $q_n^{(\nu)}(x)$:

$$q_n^{(\nu)}(x) = \frac{p_n^{(n)}(x)}{(n!)^2} + \sum_{m=0}^{n-1} \left(\frac{1}{m!}\right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu). \tag{77}$$

Via (60) this yields (9).

Let us now proceed and prove the findings related to the second conjecture [2]; since the treatment is analogous to that given above, our presentation is more terse.

The fact that the polynomials defined by (20) with (19) satisfy the recursion relation (24) is an easy consequence of the determinantal definition (20) with (19).

We now renormalize the polynomials $\tilde{p}_n^{(\nu)}(x)$ via the definition

$$\tilde{p}_n^{(\nu)}(x) = n!(n+1)!\tilde{q}_n^{(\nu)}(x), \tag{78}$$

entailing that the polynomials $\tilde{q}_n^{(\nu)}(x)$ satisfy the three-term recursion relation

$$(n+1)(n+2)\tilde{q}_{n+1}^{(\nu)}(x) = [x + 2(n+1)(n-\nu)]\, \tilde{q}_n^{(\nu)}(x) - (n-\nu)(n-1-\nu)\tilde{q}_{n-1}^{(\nu)}(x), \quad n = 1, 2, \ldots, \tag{79a}$$

$$\tilde{q}_{-1}^{(\nu)}(x) = 0, \qquad \tilde{q}_0^{(\nu)}(x) = 1, \qquad \tilde{q}_1^{(\nu)}(x) = \frac{x}{2} - \nu. \tag{79b}$$

Next, we introduce the generating function

$$\tilde{Q}(x, \nu; z) = \sum_{n=0}^{\infty} (z+1)^{-n} \tilde{q}_n^{(\nu)}(x). \tag{80}$$

It is then rather easy to verify that the recursion relation (79) entails that this generating function satisfies the second-order ODE

$$(z+1)^2 z^2 \tilde{Q}'' - 2\nu(z+1)z\tilde{Q}' + [(2\nu - x)(z+1) + \nu(\nu-1)]\, \tilde{Q} = 0. \tag{81}$$

It is also plain that, at large values of $|z|$, via (79b) we get from (80),

$$\tilde{Q}(x, \nu; z) = 1 + \frac{x - 2\nu}{2z} + O\left(\frac{1}{|z|^2}\right). \tag{82}$$

We now set

$$\tilde{Q}(x, \nu; z) = (z + 1)^{1-\nu} z^{\nu-1} \tilde{F}(x, \nu; z), \tag{83}$$

getting thereby

$$(z + 1)z^2 \tilde{F}'' - 2z\tilde{F}' + (2 - x)\tilde{F} = 0. \tag{84}$$

Hence by setting

$$\tilde{F}(x, \nu; z) = \sum_{n=0}^{\infty} \tilde{C}_n(x, \nu) z^{-n}, \tag{85}$$

we get the recursion relation

$$\tilde{C}_n(x, \nu) = \frac{x - n(n + 1)}{n(n + 1)} \tilde{C}_{n-1}(x, \nu) \tag{86a}$$

entailing

$$\tilde{C}_n(x, \nu) = \frac{\tilde{p}_n^{(n)}(x)}{n!(n + 1)!} \tilde{C}_0(x, \nu). \tag{86b}$$

Here and hereafter $\tilde{p}_n^{(n)}(x)$ is defined by (23).

Hence we conclude that a solution of (84) is provided by the formula

$$\tilde{F}(x, \nu; z) = \tilde{C}_0(x, \nu) \sum_{n=0}^{\infty} \frac{\tilde{p}_n^{(n)}(x)}{n!(n + 1)!} z^{-n}, \tag{87}$$

with $\tilde{C}_0(x, \nu)$ an arbitrary function of its two arguments. Via (83) this yields for the generating function the expression

$$\tilde{Q}(x, \nu; z) = \tilde{C}_0(x, \nu) \left(1 + \frac{1}{z}\right)^{1-\nu} \sum_{n=0}^{\infty} \frac{\tilde{p}_n^{(n)}(x)}{n!(n + 1)!} z^{-n}, \tag{88}$$

entailing at large z (via (23))

$$\tilde{Q}(x, \nu; z) = \tilde{C}_0(x, \nu) \left(1 + \frac{x - 2\nu}{2z}\right) + O\left(\frac{1}{|z|^2}\right). \tag{89}$$

Comparing with (82) we therefore conclude that $\tilde{C}_0(x, \nu) = 1$, yielding for $\tilde{Q}(x, \nu; z)$ the final expression

$$\tilde{Q}(x, \nu; z) = \left(1 + \frac{1}{z}\right)^{1-\nu} \sum_{n=0}^{\infty} \frac{\tilde{p}_n^{(n)}(x)}{n!(n + 1)!} z^{-n}. \tag{90}$$

But (80) entails

$$\tilde{q}_n^{(\nu)}(x) = (2\pi \mathrm{i})^{-1} \oint \mathrm{d}z (z + 1)^{n-1} \tilde{Q}(x, \nu; z). \tag{91}$$

Hence, via the same route as above, we obtain the following expression of the polynomials $\tilde{q}_n^{(\nu)}(x)$:

$$\tilde{q}_n^{(\nu)}(x) = \frac{\tilde{p}_n^{(n)}(x)}{n!(n + 1)!} + \sum_{m=0}^{n-1} \left\{\left[\frac{1}{m!(m + 1)!}\right] \frac{\tilde{p}_m^{(m)}(x)}{(n - m)!} \prod_{\ell=m+1}^{n} (\ell - \nu)\right\}. \tag{92}$$

Via (78) this yields (25).

Finally let us indicate, now quite tersely, how to obtain formula (46) from the three-term recursion relation (44), that follows rather obviously from the definition (40) with (39).

First of all it is convenient to perform the renormalization

$$s_n^{(\nu)}(x) = n!w_n^{(\nu)}(x), \tag{93}$$

yielding from (44) the three-term recursion relation

$$(n+1)w_{n+1}^{(\nu)}(x) = xw_n^{(\nu)}(x) + (n-\nu)w_{n-1}^{(\nu)}(x), \tag{94a}$$

with the 'initial assignments'

$$w_{-1}^{(\nu)}(x) = 0, \qquad w_0^{(\nu)}(x) = 1. \tag{94b}$$

It is then easily seen that the corresponding generating function,

$$W(x,\nu;z) = \sum_{n=0}^{\infty}(z+1)^{-n}w_n^{(\nu)}(x), \tag{95}$$

satisfies the following first-order ODE:

$$z(z+1)(z+2)W' = -\left[x(z+1)+1-\nu\right]W, \tag{96}$$

whose solution reads as follows:

$$W(x,\nu;z) = W(x,\nu)\left(1+\frac{1}{z}\right)^{1-\nu}\sum_{n=0}^{\infty}\frac{\hat{s}_n^{(\nu)}(x)}{n!z^n}, \tag{97}$$

with the polynomial $\hat{s}_n^{(\nu)}(x)$ defined by (46*b*). But (95) entails

$$w_n^{(\nu)}(x) = \oint\frac{\mathrm{d}z}{2\pi\mathrm{i}}(z+1)^{n-1}W(x,\nu;z), \tag{98}$$

yielding

$$w_n^{(\nu)}(x) = \frac{W(x,\nu)}{n!}\left\{\hat{s}_n^{(\nu)}(x) + \sum_{m=0}^{n-1}\left[\binom{n!}{m!}\frac{\hat{s}_m^{(\nu)}(x)}{(n-m)!}\prod_{\ell=m+1}^{n}(\ell-\nu)\right]\right\}. \tag{99}$$

Via the second of the 'initial data' (94*b*), together with (46*b*), one immediately sees that the *a priori* arbitrary quantity $W(x,\nu)$ must be assigned the value unity, $W(x,\nu) = 1$, and then from (93) one obtains (46*a*).

3. Outlook

Some of the results reported in this paper can be arrived at more directly by focussing on polynomials such as those considered above but having (appropriate) *integer* assignments for the parameter ν; and in this manner *Diophantine* findings analogous to those reported above can also be obtained for additional classes of orthogonal polynomials. Indeed it can be shown that the orthogonal polynomials $p_n^{(\nu)}(x)$, $\tilde{p}_n^{(\nu)}(x)$ and $s_n^{(\nu)}(x)$ discussed in this paper are included in a multi-parameter class of orthogonal polynomials which includes as well, for appropriate values of the parameters, several 'named' polynomials (Laguerre, Jacobi, Dual Hahn, Krawtchouk, Meixner, . . .), for which *Diophantine* results analogous to those reported above also hold. These findings will be reported in a separate paper [11].

Finally let us mention that Christophe Smet pointed out that the search for *integer* zeros of orthogonal polynomials is connected to the existence of perfect codes, and that in this context the results in [12] might point towards further applications of our results—a suggestion worth pursuing in future publications.

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